Objective: $\forall x \in \mathcal{X}, \forall k \in \mathcal{Y}$, $\mathbb{P}(Y=k \mid X=x)$ $\longrightarrow$ classification problem

Bayes Thm

$$\mathbb{P}(Y=k \mid X=x) = \frac{\mathbb{P}(X=x \mid Y=k)\,\mathbb{P}(Y=k)}{\sum_{c=1}^{K} \mathbb{P}(X=x \mid Y=c)\,\mathbb{P}(Y=c)} \quad (1) \approx \frac{\left[\prod_{l=1}^{m_k} \mathbb{P}(f_l^k(X)=f_l^k(x) \mid Y=k)\right]\cdot \mathbb{P}(Y=k)}{\sum_{c=1}^{K}\left\{\left[\prod_{l=1}^{m_c} \mathbb{P}(f_l^c(X)=f_l^c(x) \mid Y=c)\right]\cdot \mathbb{P}(Y=c)\right\}}$$

intractable in input space $\mathcal{X}$

Defn.

space of latent representations

$f_l^k : \mathcal{X} \to \Omega$ s.t.

$$\forall x \in \mathcal{X},\ f_l^k(x) := \underset{z \in \text{patches}(f(x))}{\arg\min} \|z - p_l^k\|_2$$

latent rep. of x

$$\begin{aligned}
\mathbb{P}(X=x \mid Y=k) &= 1 \cdot \mathbb{P}(X=x \mid Y=k) \\
&= \mathbb{P}(\underbrace{f_1^k(X)=f_1^k(x), \dots, f_{m_k}^k(X)=f_{m_k}^k(x)}_{:= F^k(X,x)} \mid X=x, Y=k)\cdot \mathbb{P}(X=x \mid Y=k) \\
&= \mathbb{P}(F^k(X,x), X=x \mid Y=k) \\
&= \mathbb{P}(X=x \mid F^k(X,x), Y=k)\,\mathbb{P}(F^k(X,x) \mid Y=k)
\end{aligned}$$

$$\because \left(\mathbb{P}(A \mid B\wedge C)\cdot \mathbb{P}(B \mid C) = \frac{\mathbb{P}(A\wedge B\wedge C)}{\cancel{\mathbb{P}(B\wedge C)}}\cdot\frac{\cancel{\mathbb{P}(B\wedge C)}}{\mathbb{P}(C)} = \mathbb{P}(A\wedge B \mid C) \right)$$

Assumption 1: $\forall x \in \mathcal{X}, \forall a,b \in [K]$, $\mathbb{P}(X=x \mid F^a(X,x), Y=a) = \mathbb{P}(X=x \mid F^b(X,x), Y=b)$

$$\Longrightarrow (1) = \frac{\cancel{\mathbb{P}(X=x \mid F^k(X,x), Y=k)}\,\mathbb{P}(F^k(X,x) \mid Y=k)\,\mathbb{P}(Y=k)}{\sum_{c=1}^{K} \cancel{\mathbb{P}(X=x \mid F^c(X,x), Y=c)}\,\mathbb{P}(F^c(X,x) \mid Y=c)\,\mathbb{P}(Y=c)}$$

$$= \frac{\mathbb{P}(F^k(X,x) \mid Y=k)\,\mathbb{P}(Y=k)}{\sum_{c=1}^{K} \mathbb{P}(F^c(X,x) \mid Y=c)\,\mathbb{P}(Y=c)} \quad (2)$$

Assumption 2: $\forall x \in \mathcal{X}$, $\mathbb{P}(F^k(X,x) \mid Y=k) = \prod_{l=1}^{m_k} \mathbb{P}(f_l^k(X)=f_l^k(x) \mid Y=k)$

$$\Longrightarrow (2) = \frac{\left[\prod_{l=1}^{m_k} \mathbb{P}(f_l^k(X)=f_l^k(x) \mid Y=k)\right]\cdot \mathbb{P}(Y=k)}{\sum_{c=1}^{K}\left\{\left[\prod_{l=1}^{m_c} \mathbb{P}(f_l^c(X)=f_l^c(x) \mid Y=c)\right]\cdot \mathbb{P}(Y=c)\right\}} \quad (3)$$

more tractable in latent space Ω

$\therefore$ with assumption 1 & 2,

$$\mathbb{P}(Y=k \mid X=x) \propto \left[\prod_{l=1}^{m_k} \mathbb{P}(f_l^k(X)=f_l^k(x) \mid Y=k)\right]\cdot \mathbb{P}(Y=k)$$

We want $\underline{\mathbb{P}(f_l^k(X)=z \mid Y=k)} = d_l^k(\|z-p_l^k\|_2)$ where $d_l^k : [0,\infty) \to [0,\infty)$ is monotonically decreasing and satisfies $\int_\Omega d_l^k(\|z-p_l^k\|_2)\,dz = 1$.

(a probability distribution)

a conditional distribution of
the probability that the random variable $f_l^k(X)$
equals z given the class Y is k.

$\longrightarrow$ enforces property of latent representations that latent representations close to prototypical latent representations are more likely.

$\longrightarrow$ we want $f_l^k(X)$ to have a distribution that satisfies this

$$\Longrightarrow (3) = \frac{\left[\prod_{l=1}^{m_k} d_l^k(\|f_l^k(x)-p_l^k\|_2)\right]\cdot \mathbb{P}(Y=k)}{\sum_{c=1}^{K}\left\{\left[\prod_{l=1}^{m_c} d_l^c(\|f_l^c(x)-p_l^c\|_2)\right]\cdot \mathbb{P}(Y=c)\right\}}$$