Duality.

def. $V' := L(V, F)$

def. $v_1, \cdots, v_n$ basis V
$\Rightarrow \phi_i(v_j) = \delta_{ij}$ basis V'

def. $U^0 := \{\phi \in V' \mid \phi(u) = 0 \ \forall u \in U\}$

propn. $\dim U^0 \overset{?}{=} \dim V - \dim U$

$U^0 = \text{null } T$; $T: V' \to U'$, $\phi \mapsto \phi|_U$

$V = U \oplus X$
for $\varphi \in U'$, define $\varphi(u+x) = \varphi(u)$

T is surj. since every linear functional $U \to F$ extends to a linear functional $V \to F$

def. $T': W' \to V'$, $\varphi \mapsto \varphi \circ T$

$M(T) = M(T')^T$ & $\dim \text{Im}\, T = \dim \text{Im}\, T'$

$\Rightarrow$ thm. col rank A = row rank A

$\dim V - \text{nullity } T \overset{?}{=} \dim W' - \text{nullity } T'$

$\text{nullity } T' = \dim \{\varphi \in W' \mid T'(\varphi) = 0\}$
$= \dim \{\varphi \in W' \mid \varphi \circ T = 0\}$
$= \dim (\text{range } T)^0$
$= \dim V' - \dim \text{range } T$
$= \dim V' - (\dim W' - \text{nullity } T)$ rank-nullity

$\Rightarrow$ BHS $= \dim V' - \text{nullity } T$
$= \dim V - \text{nullity } T$

$\dim V = \dim V'$
$(\because F^n \cong F^{n\prime})$

propn. T surj $\overset{?}{\Leftrightarrow} T'$ inj ($\Leftrightarrow$ (T inj $\Leftrightarrow T'$ surj))

LHS $\Rightarrow \forall w \in W \ \exists v \in V$ s.t. $Tv = w$
$\Rightarrow \forall \psi \in W'$ s.t. $\psi \neq 0$, $\exists w \in W$ s.t. $\psi(w) \neq 0$
$\exists v \in V$ s.t. $Tv = w$
$\Rightarrow \psi \circ T \neq 0$

T not surj $\overset{?}{\Rightarrow} T'$ not inj
$\Rightarrow \exists w \in W$ s.t. $\forall v \in V$ $Tv \neq w$
$\Rightarrow$ if $\psi(w) \neq 0$ & $\psi(w') = 0$ $\forall w' \neq w \in W$,
$\psi \circ T = 0$

range $T' \overset{?}{=} (\text{null } T)^0$

$\varphi \in$ range $T' \Rightarrow \varphi = \psi \circ T$
$\forall v \in \text{null } T$, $\varphi(v) = \psi(Tv) = \psi(0) = 0$
$\Rightarrow$ range $T' \subseteq (\text{null } T)^0$ (inclusion)

$\dim (\text{null } T)^0 = \dim V - \text{nullity } T = \text{rank } T$
$= \text{rank } T'$ (equal dim)

$F[x]$ is a Euclidean Domain w/ norm deg.

$\Longrightarrow$ division alg. well-defined (+ Bezout)

$\Downarrow$

$F[x]$ is a PID.

Assume I is an ideal in $F[x]$, $p \in I$, m is the monic polyn. of least deg. in I (WOP)

$\Longrightarrow p(x) = m(x)Q(x) + R(x)$

$R(x) = p(x) - m(x)Q(x) \in I$

$\deg R < \deg m$ $\lightning$ minimality

$\Downarrow$

$\text{null}\ \alpha = F[x] \cdot m(x)$ for some minimal polyn. $m(x)$

$F[x]$ is a UFD

FToA $\Longrightarrow$ $x - \lambda$ are the only monic irred. polyn.s if $F = \mathbb{C}$

propn. $f(\lambda) = 0 \overset{?}{\Longleftrightarrow} x - \lambda \mid f$

LHS $\Longrightarrow f(x) = (x-\lambda)Q(x) + r$ (r const.)

$f(\lambda) = 0 \Longrightarrow r = 0$

RHS $\Longrightarrow f(x) = (x-\lambda)Q(x)$

$\Longrightarrow f(\lambda) = 0$

Polynomials.

def. $F[x] \xrightarrow{\alpha} L(V)$ w.r.t. $T \in L(V)$

$f(x) \longmapsto f(T)$

Thm. $p \in F[x]$ irred. $\overset{?}{\Longrightarrow} \deg p \mid \dim \text{null}\ p(T)$

$K = F[x]/_{p(x)}$ is a field since p irred., $\dim K = \deg p$

$p(x)$ irred. $\Longrightarrow \forall f \in K$ $\gcd(f,p) = 1$

$\Longrightarrow fr + ps = 1$

$\Longrightarrow f^{-1} = r$ in K

$X = \text{null}\ p(T)$ can be thought of as a K-vector space

$\Longrightarrow \dim X = \dim K \cdot \dim_K X = \deg p \cdot \dim_K X$

$\Downarrow$

*leverage linear/quadratic nature of polyn. factors over $\mathbb{R}$

Cor.
every operator on an odd-dim $\mathbb{R}$ vector space has an eigenvalue.

induction on $\dim V$

$m(x) = p_1(x) \cdots p_t(x)$ assume $p_1, \cdots, p_t$ quadratic

if not, $\exists$ eig

by prev. thm ($\deg p \mid \text{nullity}\ p(T)$)

$2 \mid \text{nullity}\ p_1(T)$

$\Longrightarrow \text{rank}\ p_1(T)$ is odd

IH on $T|_{\text{range}\ p_1(T)}$ gives eigenvalue of T

FToA $\Longrightarrow m(x)$ has a root
$\Longleftrightarrow T$ has an eigenvector

Thm.
$m(\lambda) = 0 \overset{?}{\Longleftrightarrow} \exists v \in V$ s.t. $Tv = \lambda v$

LHS $\Longrightarrow m(x) = (x-\lambda)Q(x)$

$m(T) = (T - \lambda I)Q(T) = 0$

$\forall v \in V$ $Q(T)v = 0 \Longrightarrow$ $\lightning$ minimality

$\therefore \exists v \in V$ s.t. $(T - \lambda I)v = 0$

RHS $\Longrightarrow T^i v = \lambda^i v$

$\Longrightarrow m(T)v = a_0 v + a_1 Tv + \cdots + a_n T^n v$

$= (a_0 + a_1 T + \cdots + a_n T^n)v$

$= (a_0 + a_1\lambda + \cdots + a_n \lambda^n)v$

$= 0$

$v \neq 0 \Longrightarrow m(\lambda) = 0$

Eigenvalues & Eigenvectors

def. $Tv = \lambda v \Longrightarrow E(\lambda, T)$ is T-inv.

Thm.
$v_1, \cdots, v_n$ are distinct eigenvectors corr. to distinct eigenvalues $\lambda_1, \cdots, \lambda_n$ $\overset{?}{\Longrightarrow}$ $v_1, \cdots, v_n$ L.I.

Suppose $v_1, \cdots, v_t$ is the minimal list of L.D. eigenvectors s.t. any list shorter is L.I.

$a_1 v_1 + \cdots + a_t v_t = 0 \Longrightarrow a_1, \cdots, a_t$ all nonzero by minimality

$T(a_1 v_1 + \cdots + a_t v_t) - \lambda_t(a_1 v_1 + \cdots + a_t v_t) = 0$

$a_1(\lambda_1 - \lambda_t)v_1 + \cdots + a_{t-1}(\lambda_{t-1} - \lambda_t)v_{t-1} = 0$ $\lightning$ minimality

Propn. $\Longleftrightarrow$ eigvals of $T_{V/U} \subseteq$ eigvals of T

for finite V, $\text{null}(T_{V/U} - \lambda I) \neq 0$

$\overset{?}{\Longrightarrow} \text{null}(T - \lambda I) \neq 0$

Let $T' = T - \lambda I \Longrightarrow T'_{V/U} : v + U \longmapsto Tv - \lambda v + U$

$= T_{V/U} - \lambda I$

cp: $\text{null}\ T' = 0 \overset{?}{\Longrightarrow} \text{null}\ T'_{V/U} = 0$

LHS $\Longrightarrow T'$ bijective

*make the statement simpler via substitution

finite V

$\Longrightarrow T'|_U : U \to U$ bijective

$\Longrightarrow T'_{V/U}$ bijective

Thm.
$T \in L(V)$, V finite-dim $\overset{?}{\Longrightarrow} \deg m(x) \leq \dim V$

induction on $\dim V$.

Let $v, Tv, \cdots, T^n v$ be L.D. list of minimal length (v nonzero)

$X = \text{span}\{v, Tv, \cdots, T^{n-1}v\}$, cyclic subspace of v

$T^n v = a_0 v + a_1 Tv + \cdots + a_{n-1}T^{n-1}v \Longrightarrow p(x) = x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$

$p(T|_X) = 0$

By IH, minimal polyn. of $T_{V/X}$, $q(x)$ has $\deg q \leq \dim V/X$

$q(T) = X \Longrightarrow (pq)(T) = 0$, $\deg pq \leq \dim X + \dim V/X = \dim V$

X is T-inv

Thm.

$X_i = B_1 \oplus \cdots \oplus B_{\text{nullity } U|_{X_i}}$ where $B_i =$ cyclic subspace of some v

equiv. to $B \times Y \xrightarrow{\Sigma} X$ is bijective

$X_i = \{ v \in V \mid (T - \lambda_i I)^k v = 0 \text{ for some } k \geq 1 \} \stackrel{?}{=} B \oplus Y$ — ⓐ

Let $U|_{X_i}^m = 0$, $v \in X_i$ s.t. $U^{m-1} v \neq 0 \longrightarrow B = \operatorname{span}\{v, Uv, \cdots, U^{m-1} v\}$

(cyclic subspace)
$\dim \operatorname{span}\{v, Uv, \cdots, U^{m-1} v\} = m$

$Y \leq X_i$

$\dim Y \stackrel{①}{=} \dim X_i - \dim B \wedge B \cap Y \stackrel{②}{=} \{0\} \Longrightarrow$ ⓐ

Σ map must be bijective by inj. + dim arg.

ⓐ $\wedge$ Y $\stackrel{③}{U}$-inv $\Longrightarrow X_i = \bigoplus_i B_i$ by induction on $\dim X_i$

①: $Z = \operatorname{span}\{\varphi, U'\varphi, \cdots, U'^{m-1}\varphi\}$ where $(U'^{m-1}\varphi)(v) \neq 0$

defn of Z depends on v

$Y = Z^0 \subseteq X_i$, $\dim Y = \dim X_i - \dim Z = \dim X_i - m$

②: Assume $x \in B$ s.t. $x = a_{m-1} U^{m-1} v + \cdots + a_1 U v + a_0 v$

Assume $x \in Y = Z^0$ as well

$\Longrightarrow U'^{m-1}\varphi(x) = 0 \Longrightarrow \varphi(a_0 U^{m-1} v) = 0$

$U'^{m-1}\varphi(v) \neq 0 \Longrightarrow a_0 = 0 \Longrightarrow a_1, \cdots, a_{m-1} = 0$

③:

*Y is the annihilator of some $Z \leq X_i'$

Lemma.

$Y = Z^0 = \{ \operatorname{eval}_y \in X_i'' \mid \operatorname{eval}_y(\varphi) = 0 \ \forall \varphi \in Z \}$

$= \{ y \in X_i \mid \varphi(y) = 0 \ \forall \varphi \in Z \}$

Z U'-inv $\stackrel{?}{\Longrightarrow}$ Y U-inv

LHS $\Longrightarrow \forall \psi \in Z$, $\psi \circ U \in Z$

$Y = Z^0 \Longrightarrow \forall y \in Y$, $\psi(y) = 0$.

$\psi \circ U \in Z \Longrightarrow \psi(Uv) = 0 \Longrightarrow Uv \in Y$

Cor.

$m(x) = (x - \lambda_1)^{n_1} \cdots (x - \lambda_t)^{n_t} \iff V = X_1 \oplus \cdots \oplus X_t$

where $U_i|_{X_i} = 0$

$((T - \lambda_i I)^{n_i} = U_i)$

Jordan Form.

Propn. T diagonalizable

$\stackrel{?}{\iff} m(x) = (x - r_1) \cdots (x - r_n)$ w/ $r_1, \cdots, r_n$ distinct

LHS $\Longrightarrow \lambda_1, \cdots, \lambda_n$ diags. $(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$

$\Longrightarrow m(x) \mid (T - \lambda_1 I) \cdots (T - \lambda_n I)$

$\Longleftarrow$: induction on $\dim V$.

Let $p = (x - r_1)$, $q = m/p$.

$\gcd(p, q) = 1 \Longrightarrow V = \operatorname{null} m(T) = \operatorname{null} p(T) \oplus \operatorname{null} q(T)$

$T|_{\operatorname{null} q(T)}$ has min polyn. $q(T)$

$\operatorname{null} p(T) = E(r_1, T)$, IH $\Longrightarrow \operatorname{null} q(T) = \bigoplus_{i=2}^{n} E(r_i, T)$

$\Longrightarrow V = \bigoplus_i E(r_i, T)$

$\gcd(p, q) = 1 \stackrel{?}{\Longrightarrow} \operatorname{null} pq(T) = \operatorname{null} p(T) \oplus \operatorname{null} q(T)$

LHS $\Longrightarrow \exists r, s$ s.t. $pr(T) + qs(T) = I$

$\forall v \in V$, $pr(T)v = 0 \wedge qs(T)v = 0$ cannot be true

$\Longrightarrow \operatorname{null} p(T) \cap \operatorname{null} q(T) = \{0\}$

Diagonalizability.

Propn. T diagonalizable $\iff$

$\exists$ eigenbasis of T $\iff$

$V = \bigoplus_i E(\lambda_i, T) \iff$

$\dim \left(\bigoplus_i E(\lambda_i, T) \right) = \dim V$

Propn. given T is upper-tri.,

(eigenvalues) = (diagonal entries)

$\subseteq$: if $\lambda_1, \cdots, \lambda_n$ are on diagonal, then

$(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$

$T - \lambda_i I$ sends $\operatorname{span}\{v_1, \cdots, v_i\}$ to $\operatorname{span}\{v_1, \cdots, v_{i-1}\}$

$\Longrightarrow m(x) \mid (T - \lambda_1 I) \cdots (T - \lambda_n I)$

$\supseteq$: $(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$

$(T - \lambda_i I) : \operatorname{span}\{v_1, \cdots, v_i\} \longrightarrow \operatorname{span}\{v_1, \cdots, v_{i-1}\}$

$\Longrightarrow T - \lambda_i I$ is not inj.

④ $\Longrightarrow$ concatenate bases of each $E(\lambda_i, T)$ to obtain subspace of V w/ $\dim V$

$\Longrightarrow \bigoplus_i E(\lambda_i, T) = V$

$\Longrightarrow$ ②

② $\Longrightarrow \forall v \in V$, $v = \sum_i c_i u_i$ for eigs u_i

$\Longrightarrow$ each u_i corr. to λ_i, $V \subseteq \bigoplus_i E(\lambda_i, T)$

$\Longrightarrow V = \bigoplus_i E(\lambda_i, T)$

Lemma.

$ST = TS \overset{?}{\Longrightarrow} E(\lambda, S)$ is T-inv.

$\forall v \in E(\lambda, S)$, $Sv = \lambda v$

$STv = TSv = \lambda Tv \Longrightarrow Tv \in E(\lambda, S)$

$\Downarrow$

Thm.

S, T are diagonalizable

S, T share an eigenbasis $\overset{?}{\Longleftrightarrow} ST = TS$ $(F = \mathbb{C})$

$\Longrightarrow$: by multiplication of diagonal matrices

$\Longleftarrow$: Let $Sv = \lambda v$

$m_{T|_{E(\lambda,S)}}(x) \mid m_T(x)$

T diagonal $\Longleftrightarrow$ m_T linear factors

$E(\lambda, S)$ T-inv $\Longrightarrow T|_{E(\lambda,S)}$ has eigenbasis

concatenate such bases

Commuting Operators

def. $ST = TS$

Thm.

$\forall S, T$ s.t. $TS = ST$,

$\exists v \in V$ s.t. v is eigvec. of S and T $(F = \mathbb{C})$

By FToA, T has an eig λ

$E(\lambda, T)$ S-inv $\Longrightarrow$ by FToA, $S|_{E(\lambda,T)}$ has eigenvector $v \in E(\lambda, T)$

$\Downarrow$

Propn.

$ST = TS \overset{?}{\Longrightarrow} \exists$ basis in which S, T are both upper-tri.

$$\begin{bmatrix} \lambda & a \\ 0 & c \end{bmatrix} \begin{bmatrix} \mu & b \\ 0 & D \end{bmatrix} = \begin{bmatrix} \lambda\mu & \lambda b + aD \\ 0 & cD \end{bmatrix}$$

$$\begin{bmatrix} \mu & b \\ 0 & D \end{bmatrix} \begin{bmatrix} \lambda & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} \lambda\mu & \mu a + bc \\ 0 & Dc \end{bmatrix}$$

$\Longrightarrow cD = Dc$

$\Longrightarrow$

Prop'n.

$ST = TS \Longrightarrow$ eigs of $S+T$ are the sums of eigs of S & eigs of T

Inner Products.

Defn.

Over $\mathbb{R}$,

$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$

- symmetric, bilinear
- $\forall v \in V$, $\langle v, v \rangle \geq 0$
- $\langle v, v \rangle = 0 \iff v = 0$

Over $\mathbb{C}$,

$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$

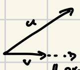
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$, conjugate-linear in second var.
- $\forall v \in V$, $\langle v, v \rangle \geq 0$ (real, non-negative)
- $\langle v, v \rangle = 0 \iff v = 0$

Thm. (C-S)

$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$, equality iff $u = cv$

$\|\lambda v\| = |\lambda| \cdot \|v\|$

idea:



$\|proj_v(u)\| \leq \|u\|$

$$\|proj_v(u)\| = \left\| \frac{\langle u, v \rangle}{\langle v, v \rangle} v \right\| = \left| \frac{\langle u, v \rangle}{\langle v, v \rangle} \right| \cdot \|v\| \leq \|u\|$$

$$\frac{|\langle u, v \rangle|}{\|v\|^2} \leq \frac{\|u\|}{\|v\|} \implies |\langle u, v \rangle| \leq \|u\| \cdot \|v\|$$

* $\alpha : V \to V'$, $v \mapsto \langle \cdot , v \rangle$

α is linear but $\alpha(v)$ is only conjugate-linear

Cor. (Δ)

$\|u + v\| \leq \|u\| + \|v\|$, equality iff $u = c'v$ where $c' \geq 0$

non-negative real

$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + \langle u, v \rangle + \langle v, u \rangle$, $(\|u\| + \|v\|)^2 = \|u\|^2 + \|v\|^2 + 2\|u\| \cdot \|v\|$

$\langle u, v \rangle \leq \|u\| \cdot \|v\|$ $\wedge$ $\langle v, u \rangle \leq \|u\| \cdot \|v\|$ $\implies$ LHS $\leq$ RHS

* Bessel's Inequality:

$e_1, \cdots, e_n$ is ON list

$\implies \forall v \in V$, $\|v\|^2 \geq |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$

* Parseval's Identity: $(T: V \to W)$

$$\sum_j \langle Te_i, Te_i \rangle_W = \sum_j \langle \langle Te_i, f_j \rangle f_j, \langle Te_i, f_j \rangle f_j \rangle_W$$

$$= \sum_j |\langle Te_i, f_j \rangle|^2$$

Propn.

allows for

injective + (dim domain = dim codomain) $\implies$ bijective

Given V is finite-dimensional, $\alpha : V \to V'$ is

an isomorphism if $F = \mathbb{R}$ and,

a conjugate-linear bijection if $F = \mathbb{C}$

* Parseval's Identity

$$= \sum_j \langle \langle Te_i, f_j \rangle f_j, \langle Te_i, f_j \rangle f_j \rangle$$

$$= \sum_j \langle Te_i, f_j \rangle \cdot \overline{\langle Te_i, f_j \rangle} = \sum_j |\langle Te_i, f_j \rangle|^2$$

$\alpha(v) = 0 \implies \alpha(v)(v) = 0$

$\implies v = 0$

Thm. (Riesz Rep.)

Given V is f.d., if $\varphi \in V'$,

$\exists!\, v \in V$ s.t. $\varphi(x) = \langle x, v \rangle$

* $e_1, \cdots, e_n$ ON basis

$\implies v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$

Propn. (Schur's Thm)

If $T \in \mathcal{L}(V)$ is upper-triangular in some basis, it is upper-triangular in some ON basis.

Defn (orthogonal complement)

$U \subseteq V$ (subset)

$U^\perp := \{v \in V \mid \langle v, u \rangle = 0 \text{ for all } u \in U\} = \{v \in V \mid \varphi_v \in U^0\}$

$\{\varphi \in V' \mid \varphi(u) = 0 \text{ for all } u \in U\}$

Riesz

* $U \subseteq V \implies V^\perp \subseteq U^\perp$

Lemma.

If $U \subseteq V$ is f.d., then $V = U \oplus U^{\perp}$

$e_1, \dots, e_n$ ON basis of U

$\implies \forall v \in V,\ \ \mathrm{proj}_U(v) = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$

$\langle v - \mathrm{proj}_U(v), e_k \rangle = 0$ for all e_k $\implies v = \overset{U}{\underset{\cup}{\mathrm{proj}_U(v)}} + \overset{U^{\perp}}{\underset{\cup}{(v - \mathrm{proj}_U(v))}}$

$(+\ U \cap U^{\perp} \subseteq \{0\})$

* $U \subseteq V$, U f.d. $\forall u \in U$,

$\|v - P_U(v)\| \leq \|v - u\|$

$\|v - P_U(v)\|^2 \leq \|v - P_U(v)\|^2 + \|P_U(v) - u\|^2$ Pythagoras

$= \|v - P_U(v) + P_U(v) - u\|^2$

$= \|v - u\|^2$

Thm.

no restriction on V

Given only that $U \subseteq V$ is f.d., $U = (U^{\perp})^{\perp}$.

Suppose $v \in V$ is s.t. $v \perp U^{\perp}$ (i.e. $v \in (U^{\perp})^{\perp}$)

$V = U \oplus U^{\perp} \implies v = u + w$ where $u \in U$, $w \in U^{\perp}$

$v \perp U^{\perp} \implies \langle v, w \rangle = \langle u + w, w \rangle = \langle w, w \rangle = 0$

$\implies w = 0 \implies v \in U$

Defn.

projection: $P \in \mathcal{L}(V)$ s.t. $P^2 = P$

$\exists U \subseteq V$ s.t. $V = U \oplus W$, $P = \mathrm{proj}_U$

orthogonal projection: $P \in \mathcal{L}(V)$ s.t. $P^2 = P$, $P^* = P$

$\exists U \subseteq V$ s.t. $V = U \oplus U^{\perp}$, $P = \mathrm{proj}_U$

null $P \perp$ range $P \implies P^* = P$

*
$$\begin{tikzcd}
V \arrow[r, "\alpha_V", "\sim"'] \arrow[d, "T"] & V' \\
W \arrow[r, "\alpha_W", "\sim"'] \arrow[u, bend left=60, "T^*"] & W' \arrow[u, "T'"']
\end{tikzcd}$$

$T^* := \alpha_V^{-1} T' \alpha_W$

$\implies \alpha_V T^* = T' \alpha_W$

$\implies \alpha_V T^* w = \langle \cdot, T^* w \rangle_V$

$T' \alpha_W(w) = \langle T(\cdot), w \rangle_W$

$\langle v, T^* w \rangle_V = \langle T v, w \rangle_W$

Propn.

Given V is f.d., $(T^*)^*$.

By defn, $(T^*)^*$ is the unique operator s.t.

$\langle v, (T^*)^* w \rangle_V = \langle T^* v, w \rangle_W$

$\langle (T^*)^* w, v \rangle_V = \langle w, T^* v \rangle_W$

$= \langle T w, v \rangle_W$ $\implies (T^*)^* = T$

Propn.

$M(T) = (a_{ij}) \implies M(T^*) = (\overline{a_{ji}})$

$T v_j = \sum_{i=1}^{n} a_{ij} w_i \implies a_{ij} = \langle T v_j, w_i \rangle_W$

$= \langle v_j, T^* w_i \rangle_V$

$\implies \langle T^* w_i, v_j \rangle_V = \overline{a_{ji}}$

Propn.

a vector is 0 iff perp to all vectors in its space

null $T^* = (\text{range } T)^{\perp}$

$T^* v = 0 \implies \forall w \in V,\ \langle T^* v, w \rangle = 0$

$\implies \forall w \in V,\ \langle v, T w \rangle = 0$

Spectral Theorem(s)

Thm.

$T=T^* \Longrightarrow$ all roots of $m(x)$ are real

eigs

$\langle Tv, v\rangle = \lambda \langle v, v\rangle$

$= \langle v, T^* v\rangle = \langle v, Tv\rangle = \bar{\lambda}\langle v, v\rangle$

$v \neq 0 \Longrightarrow \lambda = \bar{\lambda}$

(eigs are nonzero)

Lemma.

$F=\mathbb{C}, \ \forall v\in V, \ \langle Tv, v\rangle = 0 \iff T=0$

not true if $F=\mathbb{R}$

Cor.

$F=\mathbb{C}, \ \forall v\in V, \ \langle Tv, v\rangle \in \mathbb{R} \iff T=T^*$

$\langle Tv, v\rangle = \langle v, T^* v\rangle = \langle T^* v, v\rangle$

$\iff \langle (T-T^*)v, v\rangle = 0 \iff T-T^* = 0$

Propn.

$T=T^*$ s.t. $\langle Tv, v\rangle = 0$ for all $v \Longrightarrow T=0$

True for $F=\mathbb{C}$, so assume $F=\mathbb{R}$

$\psi: V\times V \to F, \ (v,w) \mapsto \langle Tv, w\rangle$

(bilinear)

$\psi(v,w) = \psi(w,v)$

$\psi(v+w, v+w) = 0$

$= \cancel{\psi(v,v)} + \psi(v,w) + \psi(w,v) + \cancel{\psi(w,w)}$

$= 2\psi(v,w) = 0$

symmetric

$\updownarrow$ ↯ clashing properties

$\psi(v,v)=0 \Longrightarrow$ 'alternating', i.e. ψ is antisymmetric,

i.e. $\psi(v,w) = -\psi(w,v)$

$*$ works for any bilinear function as long as division by 2 is allowed

Propn.

$T=T^*, \ b^2-4c<0 \Longrightarrow T^2+bT+cI$ is injective

$\langle (T^2+bT+cI)v, v\rangle = \langle T^2 v, v\rangle + b\langle Tv, v\rangle + c\langle v, v\rangle$

$(v\neq 0)$

$= \|Tv\|^2 + b\langle Tv, v\rangle + c\|v\|^2 \geq \|Tv\|^2 - |b|\,\|Tv\|\,\|v\| + c\|v\|^2$

reverse C-S

$= \left(\|Tv\| - \frac{|b|\,\|v\|}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)\|v\|^2 > 0$

Propn.

$T=T^* \Longrightarrow T$ has an eigenvalue

$m(x) = g_1(x)\cdots g_k(x)$ where each g_i is an irred. polyn.

if any g_i has deg 1, $\exists$ eig

else, $g_i(T)$ injective $\Longrightarrow m(T)\neq 0$ ↯

Thm.

true for $F=\mathbb{R}$ as well since T^*T-TT^* is self-adj

$$T^*T=TT^* \iff \|Tv\|=\|T^*v\| \text{ for all } v\in V$$

LHS $\iff \forall v\in V$ $\langle (T^*T-TT^*)v, v\rangle = 0$

$\iff \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle$

assume V f.d.

weaker condition than $T=T^*$, necessary for $F=\mathbb{R}$

Thm. (Complex Spectral Thm.)

If T is normal & $F=\mathbb{C}$, $\exists$ ON eigenbasis

$F=\mathbb{C} \implies T$ is upper-tri in some ON basis $e_1,\cdots,e_n$ (Schur)

$Te_1 = a_{11}e_1$, $T^*e_1 = \bar{a}_{11}e_1 + \bar{a}_{12}e_2 + \cdots + \bar{a}_{1n}e_n$ (T^* is conjugate-transpose of T)

$\|Te_1\|^2 = |a_{11}|^2 = \|T^*e_1\|^2 = |\bar{a}_{11}|^2 + \sum_{i=2}^{n}|\bar{a}_{1i}|^2$

$\implies \sum_{i=2}^{n}|a_{1i}|^2 = 0 \implies a_{12}=\cdots=a_{1n}=0$

$$\begin{bmatrix} a_{11} & \overset{0}{\cancel{a_{12} \cdots a_{1n}}} & \\ & a_{22} & 0\text{'s} \\ 0\text{'s} & \ddots & a_{nn} \end{bmatrix}$$

$\vdots$

Thm. (Real Spectral Thm)

If T is self-adj and $F=\mathbb{R}$, $\exists$ ON eigenbasis

hypothesis $\implies m(x) = \prod_i (x-\lambda_i)$

$\implies T$ upper-tri in some ON basis

$T=T^* \implies$ diagonal

Cor.

λI is normal

T normal $\implies T, T^*$ have the same eigenvectors

$Tv=\lambda v \implies \|Tv\| = |\lambda|\,\|v\| = \|T^*v\|$

$T-\lambda I$ normal $\implies \|(T-\lambda I)v\| = \|(T^*-\bar{\lambda}I)v\|$

$\implies (T-\lambda I)v=0 \iff (T^*-\bar{\lambda}I)v=0$

Cor.

T normal $\implies$ eigenvectors corr. to diff eigenvalues are orthogonal

$Tv=\lambda v$, $Tw=\mu v$

$\langle Tv, w\rangle = \lambda\langle v,w\rangle = \langle v, T^*w\rangle = \langle v, \bar{\mu}w\rangle = \mu\langle v,w\rangle$

$\implies \langle v,w\rangle(\lambda-\mu)=0$

Positive Ops & Isometries

$*$ $T \in L(V) \sim z \in \mathbb{C}$

T^*T is positive, $z\bar{z} \in \mathbb{R}$

$(T^* \sim \bar{z})$

Defn.

T pos. $\iff T = T^* \wedge \forall v \in V, \langle Tv, v \rangle \geq 0$ — real, non-negative $\to$ if $F = \mathbb{C}$, $\langle Tv, v\rangle \in \mathbb{R}$ implies $T = T^*$

TFAE

T positive $\iff T = T^*$, T has nonnegative eigs
$\iff T = R^2$ where R is positive
$\iff T = R^2$ where R is self-adj
$\iff T = R^*R$

T positive, $Tv = \lambda v \implies \langle Tv, v\rangle = \lambda\langle v, v\rangle \geq 0$
$\implies \lambda \geq 0$ since $\langle v, v\rangle > 0$

$T = T^*$, nonneg. eigs $\implies$ spectral thm
$\implies T = \mathrm{diag}(d_1, \dots, d_n)$ w/ nonneg d_i's since eigs ≥ 0
$\implies \sqrt{T} = \mathrm{diag}(\sqrt{d_1}, \dots, \sqrt{d_n})$

$T = R^*R \implies \langle Tv, v\rangle = \langle R^*Rv, v\rangle = \langle Rv, Rv\rangle \geq 0$

Thm.

T positive $\implies \exists! R \in L(V)$ s.t. R pos, $T = R^2$

need only show that for ON eigenbasis $v_1, \dots, v_n$ of T,

$Tv_i = \lambda_i v_i \implies Rv_i = \sqrt{\lambda_i} v_i \implies R$ is uniquely determined

Suppose $Tv = \lambda v$, $f_1, \dots, f_n$ ON eigenbasis of R

$v = a_1 f_1 + \dots + a_n f_n$

$Rf_i = \mu_i f_i$

$\lambda v = Tv = R^2 v = a_1 \mu_1^2 f_1 + \dots + a_n \mu_n^2 f_n$

$= a_1 \lambda f_1 + \dots + a_n \lambda f_n$

For nonzero a_i, $\lambda = \mu_i^2$

$*$ isometry operators on $\mathbb{R}/\mathbb{C}$ form orthogonal / unitary groups

Defn.

$S \in L(V, W)$ is an isometry if $\forall v \in V$, $\|Sv\| = \|v\|$

TFAE.

S is an isometry $\iff \forall u, v \in V$, $\langle Su, Sv\rangle = \langle u, v\rangle$
$\iff (e_1, \dots, e_n$ ON basis $\implies Se_1, \dots, Se_n$ ON list$)$
$\iff \exists e_1, \dots, e_n$ ON basis s.t. $Se_1, \dots, Se_n$ ON list

S is unitary $\iff S^*S = I \iff SS^* = I \iff S^*$ is unitary
$\iff S^* = S^{-1}$

Propn.

S unitary $\iff$ eigs of $S = \pm 1$

$S^*S = I$, $S = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

Recall our analogy between $\mathbf{C}$ and $\mathcal{L}(V)$. Under this analogy, a complex number z corresponds to an operator $S \in \mathcal{L}(V)$, and $\bar{z}$ corresponds to S^*. The real numbers $(z = \bar{z})$ correspond to the self-adjoint operators $(S = S^*)$, and the nonnegative numbers correspond to the (badly named) positive operators.

Another distinguished subset of $\mathbf{C}$ is the unit circle, which consists of the complex numbers z such that $|z| = 1$. The condition $|z| = 1$ is equivalent to the condition $\bar{z}z = 1$. Under our analogy, this corresponds to the condition $S^*S = I$, which is equivalent to S being a unitary operator. Hence the analogy shows that the unit circle in $\mathbf{C}$ corresponds to the set of unitary operators. In the next two results, this analogy appears in the eigenvalues of unitary operators. Also see Exercise 15 for another example of this analogy.

Polar Decomp, SVD

Analogy.

In $\mathbb{R}^2$, $(x,y) = (r\cos\theta, r\sin\theta) \rightsquigarrow (r,\theta)$

$$= \underbrace{(\cos\theta, \sin\theta)}_{\text{point on unit circle}} \cdot \underbrace{r}_{\text{positive real number}}$$

If $z \in \mathbb{C}$, $z \neq 0$

$$z = \underbrace{\frac{z}{|z|}}_{\text{point on complex unit circle}} \underbrace{|z|}_{\text{positive real number } (|z| = \sqrt{z\bar{z}})} = \frac{z}{|z|}\sqrt{\bar{z}z}$$

$\rightsquigarrow$ If $T \in L(V)$,

$$T = \underbrace{S}_{\text{isometry (unitary)}} \circ \underbrace{\sqrt{T^*T}}_{\text{positive}}$$

* unitary ops $\sim$ point on unit circle

* for any $T \in L(V,W)$,

- T^*T is positive
- null T^*T = null T
- range T^*T = range T^*
- rank T = rank T^*

$v \in \text{null } T^*T \Rightarrow \underbrace{\langle T^*Tv, v\rangle}_{= \|Tv\|^2} = 0 \Rightarrow Tv = 0$

T^*T self-adj $\Rightarrow$ range $T^*T = (\text{null } T^*T)^\perp$

range $T^* = (\text{null } T)^\perp$

Defn. (Singular Values)

The singular values of T are the eigenvalues of $\sqrt{T^*T}$ in decreasing order w/ possible repetition (λ is listed dim $E(\lambda, T)$ times)

list of eigenvalues	list of singular values
context: vector spaces	context: inner product spaces
defined only for linear maps from a vector space to itself	defined for linear maps from an inner product space to a possibly different inner product space
can be arbitrary real numbers (if $\mathbf{F} = \mathbf{R}$) or complex numbers (if $\mathbf{F} = \mathbf{C}$)	are nonnegative numbers
can be the empty list if $\mathbf{F} = \mathbf{R}$	length of list equals dimension of domain
includes 0 $\Longleftrightarrow$ operator is not invertible	includes 0 $\Longleftrightarrow$ linear map is not injective
no standard order, especially if $\mathbf{F} = \mathbf{C}$	always listed in decreasing order

Thm. (SVD)

$s_1, \ldots, s_m$ are positive singular vals. of $T \in L(V,W)$

$\exists$ ON lists $e_1, \ldots, e_m \in V$, $f_1, \ldots, f_m \in W$ s.t.

$\forall v \in V$, $Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots +$